

The Curie-Weiss Model of SOC in Higher Dimension

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Abstract

We build and study a multidimensional version of the Curie-Weiss model of self-organized criticality we have designed in [2]. For symmetric distributions satisfying some integrability condition, we prove that the sum S_n of the random vectors in the model has a typical critical behaviour. The fluctuations are of order $n^{3/4}$ and the limiting law has a density proportional to the exponential of a fourth-degree polynomial.

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1 Introduction

In [2] and [5], we introduced a *Curie-Weiss model of self-organized criticality* (SOC): we transformed the distribution associated to the generalized Ising Curie-Weiss model by implementing an automatic control of the inverse temperature which forces the model to evolve towards a critical state. It is the model given by an infinite triangular array of real-valued random variables $(X_n^k)_{1 \leq k \leq n}$ such that, for all $n \geq 1$, (X_n^1, \dots, X_n^n) has the distribution

$$\frac{1}{Z_n} \exp \left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2} \right) \mathbb{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i),$$

where ρ is a probability measure on \mathbb{R} which is not the Dirac mass at 0, and where Z_n is the normalization constant. We extended the study of this model in [6], [7], [8] and [9]. For symmetric distributions satisfying some exponential moments condition, we proved that the sum S_n of the random variables behaves as in the typical critical generalized Ising Curie-Weiss model: the fluctuations are of order $n^{3/4}$ and the limiting law is $C \exp(-\lambda x^4) dx$ where C and λ are suitable positive constants. Moreover, by construction, the model does not depend on any external parameter. That is why we can conclude it exhibits the phenomenon of self-organized criticality (SOC). Our motivations for studying such a model are detailed in [2].

Let $d \geq 1$. In this paper we define a d -dimensional version of the Curie-Weiss model of SOC, i.e, such that the X_n^k , $1 \leq k \leq n$, are random vectors in \mathbb{R}^d . Let us start by defining the *d-dimensional generalized Ising Curie-Weiss model*. Let ρ be a symmetric probability measure on \mathbb{R}^d such that

$$\forall v \geq 0 \quad \int_{\mathbb{R}^d} \exp(v\|z\|^2) d\rho(z) < \infty.$$

Assume that its covariance matrix

$$\Sigma = \int_{\mathbb{R}^d} z {}^t z d\rho(z)$$

is invertible (i.e, ρ is non-degenerate on \mathbb{R}^d). The d -dimensional generalized Ising Curie-Weiss model associated to ρ and to the temperature field T (which is here a $d \times d$ symmetric positive definite matrix) is defined through an infinite triangular array of random vectors $(X_n^k)_{1 \leq k \leq n}$ such that, for all $n \geq 1$, (X_n^1, \dots, X_n^n) has the distribution

$$\frac{1}{Z_n(T)} \exp \left(\frac{1}{2n} \langle T^{-1}(x_1 + \dots + x_n), (x_1 + \dots + x_n) \rangle \right) \prod_{i=1}^n d\rho(x_i),$$

where $Z_n(T)$ is a normalization. When $d = 1$ and $\rho = (\delta_{-1} + \delta_1)/2$, we recover the classical Ising Curie-Weiss model. Let $S_n = X_n^1 + \dots + X_n^n$ for any $n \geq 1$. By extending the methods of Ellis and Newmann (see [4]) to the higher dimension, we obtain that, under some « sub-Gaussian » hypothesis on ρ , if $T - \Sigma$ is a symmetric positive definite matrix, then

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}_d(0, T(T - \Sigma)^{-1}\Sigma),$$

the centered d -dimensional Gaussian distribution with covariance matrix $T(T - \Sigma)^{-1}\Sigma$. If $T = \Sigma$ (critical case) then

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} C_\rho \exp(-\varphi_\rho(s_1, \dots, s_d)) ds_1 \dots ds_d,$$

where C_ρ is a normalization constant and φ_ρ is an homogeneous polynomial of degree four in $\mathbb{R}[X_1, \dots, X_d]$ such that $\exp(-\varphi_\rho)$ is integrable with respect to the Lebesgue measure on \mathbb{R}^d . Detailed proofs of these results are given in section 23 of [8]. These results highlight that the non-critical fluctuations are normal (in the Gaussian sense) while the critical fluctuations are of order $n^{3/4}$ (or eventually $n^{1-1/2k}$, $k \geq 3$, in the degenerate cases of sub-Gaussian measures, see [4]).

Now we try to modify this model in order to construct a d -dimensional SOC model. As in [2], we search an automatic control of the temperature field T , which would be a function of the random variables in the model, so that, when n goes to $+\infty$, T converges towards the critical value Σ of the model. We start with the following observation: if $(Y_n)_{n \geq 1}$ is a sequence of independent random vectors with identical distribution ρ , then, by the law of large numbers,

$$\frac{\hat{\Sigma}_n}{n} \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \Sigma,$$

where

$$\forall n \geq 1 \quad \widehat{\Sigma}_n = X_n^1 t(X_n^1) + \cdots + X_n^n t(X_n^n).$$

This convergence provides us with an estimator of Σ . If we believe that a similar convergence holds in the d -dimensional generalized Ising Curie-Weiss model, then we are tempted to « replace T by $\widehat{\Sigma}_n/n$ » in the previous distribution. Hence, in this paper, we consider the following model:

The model. Let $(X_n^k)_{n \geq d, 1 \leq k \leq n}$ be an infinite triangular array of random vectors in \mathbb{R}^d such that, for any $n \geq d$, (X_n^1, \dots, X_n^n) has the distribution $\tilde{\mu}_{n,\rho}$, the probability measure on $(\mathbb{R}^d)^n$ with density

$$(x_1, \dots, x_n) \mapsto \frac{1}{Z_n} \exp \left(\frac{1}{2} \left\langle \left(\sum_{i=1}^n x_i t x_i \right)^{-1} \left(\sum_{i=1}^n x_i \right), \left(\sum_{i=1}^n x_i \right) \right\rangle \right)$$

with respect to $\rho^{\otimes n}$ on the set

$$D_n^+ = \left\{ (x_1, \dots, x_n) \in (\mathbb{R}^d)^n : \det \left(\sum_{i=1}^n x_i t x_i \right) > 0 \right\},$$

where

$$Z_n = \int_{D_n^+} \exp \left(\frac{1}{2} \left\langle \left(\sum_{i=1}^n x_i t x_i \right)^{-1} \left(\sum_{i=1}^n x_i \right), \left(\sum_{i=1}^n x_i \right) \right\rangle \right) \prod_{i=1}^n d\rho(x_i).$$

For any $n \geq d$, we denote $S_n = X_n^1 + \cdots + X_n^n \in \mathbb{R}^d$ and

$$T_n = X_n^1 t(X_n^1) + \cdots + X_n^n t(X_n^n).$$

In section 2.b), we prove rigorously that this model is well-defined, i.e., $Z_n \in]0, +\infty[$ for any $n \geq d$.

According to the construction of this model and according to our results in one dimension, we expect that the fluctuations are of order $n^{3/4}$. Our main theorem states that they are indeed:

Theorem 1. *Let ρ be a symmetric probability measure on \mathbb{R}^d . Suppose that*

$$\exists v_0 > 0 \quad \int_{\mathbb{R}^d} \exp(v_0 \|z\|^2) d\rho(z) < \infty \quad (*)$$

and that the ρ -measure of any vector hyperplane of \mathbb{R}^d is less than $1/\sqrt{e}$. Let Σ be the covariance matrix of ρ and let M_4 be the function defined on \mathbb{R}^d by

$$\forall z \in \mathbb{R}^d \quad M_4(z) = \sum_{1 \leq i,j,k,l \leq d} \left(\int_{\mathbb{R}^d} y_i y_j y_k y_l d\rho(y) \right) z_i z_j z_k z_l.$$

Law of large numbers: Under $\tilde{\mu}_{n,\rho}$, $(S_n/n, T_n/n)$ converges in probability to $(0, \Sigma)$.

Fluctuation result: Under $\tilde{\mu}_{n,\rho}$,

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{\exp \left(-\frac{1}{12} M_4(\Sigma^{-1} z) \right) dz}{\int_{\mathbb{R}^d} \exp \left(-\frac{1}{12} M_4(\Sigma^{-1} u) \right) du}.$$

We prove that the matrix Σ is invertible in subsection 2.a). After giving large deviations results in subsection 2.c), we show the law of large numbers in section 3. Finally, in section 4, we prove that the function

$$z \mapsto \exp \left(-M_4 \left(\Sigma^{-1/2} z \right) / 12 \right)$$

is integrable on \mathbb{R}^d and that $S_n/n^{3/4}$ converges in distribution to the announced limiting distribution.

Remark : in the case where $d = 1$, we have already proved this theorem in [2], [7] and [9]. Moreover we succeeded to remove the hypothesis on the ρ -measure of the vector hyperplanes (which turns out to be simply $\rho(\{0\}) < 1/\sqrt{e}$ when $d = 1$) with a conditioning argument. It seems not immediate that such arguments could extend in the case where $d \geq 2$. However this assumption together with condition (*) are technical hypothesis and we believe that the result should be true if ρ is only a non-degenerate symmetric probability measure on \mathbb{R}^d having a finite fourth moment.

2 Preliminaries

In this section, we suppose that ρ is a symmetric probability measure on \mathbb{R}^d satisfying (*) and such that the ρ -measure of any vector hyperplane of \mathbb{R}^d is less than $1/\sqrt{e}$.

a) Σ is a symmetric positive definite matrix

Since ρ satisfies condition (*), the covariance matrix Σ is well-defined. It is of course a symmetric positive semi-definite matrix. Let \mathcal{H} be an hyperplane of \mathbb{R}^d . If \mathcal{H} is a vector hyperplane then, by hypothesis, $\rho(\mathcal{H}) < 1/\sqrt{e} < 1$. If \mathcal{H} is an affine (but not vector) hyperplane then,

$$\rho(\mathcal{H}) = \rho(-\mathcal{H}) = \frac{1}{2}(\rho(\mathcal{H}) + \rho(-\mathcal{H})) \leq \frac{1}{2} < 1,$$

since ρ is symmetric and $\mathcal{H} \cap (-\mathcal{H}) = \emptyset$. In both cases $\rho(\mathcal{H}) < 1$ thus ρ is a non-degenerate probability measure on \mathbb{R}^d . As a consequence Σ is positive definite (see lemma III.7 of [8]).

b) The model is well-defined

Let us prove that the model is well defined, i.e., $Z_n \in]0, +\infty[$ for any $n \geq d$.

Lemma 2. *Let $n \geq 1$ and let x_1, \dots, x_n be vectors in \mathbb{R}^d . We denote*

$$A_n = x_1 {}^t x_1 + \dots + x_n {}^t x_n.$$

- ★ *If $n < d$, then A_n is non-invertible.*
- ★ *If $n = d$, then A_n is invertible if and only if (x_1, \dots, x_n) is a basis of \mathbb{R}^d .*
- ★ *If $n > d$ and if the vectors x_1, \dots, x_n span \mathbb{R}^d , then A_n is invertible.*

Proof. \star Let $n \leq d$. If $n < d$, we put $x_{n+1} = \dots = x_d = 0$. We denote by B the $d \times d$ matrix such that its columns are x_1, \dots, x_d . We have then, for any $1 \leq k, l \leq d$,

$$(B^t B)_{k,l} = \sum_{i=1}^d B_{k,i} B_{l,i} = \sum_{i=1}^d x_i(k) x_i(l) = \sum_{i=1}^d (x_i^t x_i)_{k,l} = (A_n)_{k,l}.$$

Therefore $A_n = B^t B$ and thus A_n is invertible if and only if B is invertible. As a consequence A_n is invertible if and only if (x_1, \dots, x_d) is a basis of \mathbb{R}^d . In the case where $n < d$, B has at least a null column and thus is not invertible.

\star Let $n > d$ and assume that the vectors x_1, \dots, x_n span \mathbb{R}^d . There exists then $1 \leq i_1 < \dots < i_d \leq n$ such that $(x_{i_1}, \dots, x_{i_d})$ is a basis of \mathbb{R}^d . As a consequence, by the previous case, A_n is the sum of a symmetric positive definite matrix and $n - d$ other symmetric positive semi-definite matrices. Therefore A_n is definite thus invertible. \square

Let $n \geq d$. The non-degeneracy of ρ implies that its support is not included in a hyperplane of \mathbb{R}^d . As a consequence

$$\rho^{\otimes n}(\{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : (x_1, \dots, x_d) \text{ is a basis of } \mathbb{R}^d\}) > 0.$$

The previous lemma yields

$$\rho^{\otimes n}(\{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : x_1^t x_1 + \dots + x_n^t x_n \text{ is invertible}\}) > 0,$$

i.e., $\rho^{\otimes n}(D_n^+) > 0$. Therefore $Z_n > 0$.

We denote:

- \mathcal{S}_d the space of $d \times d$ symmetric matrices.
- \mathcal{S}_d^+ the space of all matrices in \mathcal{S}_d which are positive semi-definite.
- \mathcal{S}_d^{++} the space of all matrices in \mathcal{S}_d which are positive definite.

We introduce the sets

$$\Delta = \{(x, M) \in \mathbb{R}^d \times \mathcal{S}_d^+ : M - x^t x \in \mathcal{S}_d^+\}.$$

and

$$\Delta^* = \{(x, M) \in \mathbb{R}^d \times \mathcal{S}_d^{++} : M - x^t x \in \mathcal{S}_d^+\}.$$

The two following lemma guarantee that $Z_n < +\infty$ pour tout $n \geq 1$.

Lemma 3. *If $(x, M) \in \Delta^*$ then $\langle M^{-1}x, x \rangle \leq 1$.*

Proof. The matrix $M - x^t x$ is symmetric positive semi-definite. Hence

$$\forall y \in \mathbb{R}^d \quad \langle x, y \rangle^2 = \langle x^t x y, y \rangle \leq \langle M y, y \rangle.$$

Applying this inequality to $y = M^{-1}x$, we get

$$\langle x, M^{-1}x \rangle^2 \leq \langle M^{-1}x, x \rangle.$$

If $x = 0$ then $\langle M^{-1}x, x \rangle = 0 \leq 1$. If $x \neq 0$, since $M \in \mathcal{S}^{++}$, we have $\langle M^{-1}x, x \rangle > 0$ and thus $\langle M^{-1}x, x \rangle \leq 1$. \square

Lemma 4. For any vectors x_1, \dots, x_n in \mathbb{R}^d and for any positive real numbers $\lambda_1, \dots, \lambda_p$ whose sum is equal to 1, we have

$$\sum_{i=1}^p \lambda_i (x_i, x_i {}^t x_i) \in \Delta.$$

Proof. We denote

$$\nu = \sum_{i=1}^n \lambda_i \delta_{x_i}.$$

This is a probability measure on \mathbb{R}^d satisfying

$$m = \int_{\mathbb{R}^d} z d\nu(z) = \sum_{i=1}^p \lambda_i x_i \quad \text{and} \quad S = \int_{\mathbb{R}^d} z {}^t z d\nu(z) = \sum_{i=1}^p \lambda_i x_i {}^t x_i \in \mathcal{S}_d^+.$$

Hence

$$S - m {}^t m = \int_{\mathbb{R}^d} (z - m) {}^t (z - m) d\nu(z) \in \mathcal{S}_d^+,$$

proving the lemma. \square

Let $n \geq 1$. These lemma imply that, for any $(x_1, \dots, x_n) \in D_n^+$,

$$\left(\frac{1}{n} \sum_{i=1}^n x_i, \frac{1}{n} \sum_{i=1}^n x_i {}^t x_i \right) \in \Delta^*$$

and thus

$$\frac{1}{2} \left\langle \left(\sum_{i=1}^n x_i {}^t x_i \right)^{-1} \left(\sum_{i=1}^n x_i \right), \left(\sum_{i=1}^n x_i \right) \right\rangle \leq \frac{n}{2}.$$

Hence $Z_n \leq e^{n/2} < +\infty$ and the model is well-defined for any $n \geq d$.

c) Large deviations for $(S_n/n, T_n/n)$

As in the one-dimensional case (see [2]), we introduce

$$F : (x, M) \in \Delta^* \mapsto \frac{\langle M^{-1} x, x \rangle}{2}.$$

For any $n \geq d$, the distribution of $(S_n/n, T_n/n)$ under $\tilde{\mu}_{n,\rho}$ is

$$\frac{\exp(nF(x, M)) \mathbb{1}_{\{(x, M) \in \Delta^*\}} d\tilde{\nu}_{n,\rho}(x, M)}{\int_{\Delta^*} \exp(nF(s, N)) d\tilde{\nu}_{n,\rho}(s, N)},$$

where $\tilde{\nu}_{n,\rho}$ is the law of

$$\left(\frac{S_n}{n}, \frac{T_n}{n} \right) = \frac{1}{n} \sum_{i=1}^n (Y_i, Y_i {}^t Y_i)$$

when Y_1, \dots, Y_n are independent random vectors with common law ρ .

We denote by $\langle \cdot, \cdot \rangle$ the usual scalar product on \mathbb{R}^d and by $\| \cdot \|$ the Euclidean norm. We endow $\mathbb{R}^d \times \mathcal{S}_d$ with the scalar product given by

$$((x, M), (y, N)) \mapsto \langle x, y \rangle + \text{tr}(MN) = \sum_{i=1}^d x_i y_i + \sum_{i=1}^d \sum_{j=1}^d m_{i,j} n_{i,j}.$$

We denote by $\| \cdot \|_d$ the associated norm. Notice that

$$\forall z \in \mathbb{R}^d \quad \forall A \in \mathcal{S}_d \quad \text{tr}(z^t z A) = \sum_{i=1}^d \sum_{j=1}^d z_i z_j a_{i,j} = \langle Az, z \rangle.$$

Let ν_ρ be the law of $(Z, Z^t Z)$ when Z is a random vector with distribution ρ . We define its Log-Laplace Λ , by

$$\begin{aligned} \forall (u, A) \in \mathbb{R}^d \times \mathcal{S}_d \quad \Lambda(u, A) &= \ln \int_{\mathbb{R}^d \times \mathcal{S}_d} \exp(\langle z, u \rangle + \text{tr}(MA)) \, d\nu_\rho(z, M) \\ &= \ln \int_{\mathbb{R}^d} \exp(\langle u, z \rangle + \langle Az, z \rangle) \, d\rho(z), \end{aligned}$$

and its Cramér transform I by

$$\forall (x, M) \in \mathbb{R}^d \times \mathcal{S}_d \quad I(x, M) = \sup_{(u, A) \in \mathbb{R}^d \times \mathcal{S}_d} (\langle x, u \rangle + \text{tr}(MA) - \Lambda(u, A)).$$

Let D_Λ and D_I be the domains of $\mathbb{R}^d \times \mathcal{S}_d$ where Λ and I are respectively finite. All these definitions generalize the case where $d = 1$, treated in [2] and [7].

For any $(u, A) \in \mathbb{R}^d \times \mathcal{S}_d$, we have

$$\begin{aligned} \exp \Lambda(u, A) &\leq \int_{\mathbb{R}^d} \exp \left(\|u\| \|z\| + \sqrt{\text{tr}(M^2)} \|z\|^2 \right) d\rho(z) \\ &\leq \exp(\|(u, M)\|_d) + \int_{\mathbb{R}^d} \exp(\|(u, M)\|_d \|z\|^2) d\rho(z). \end{aligned}$$

Therefore condition $(*)$ is sufficient to ensure that $(0, O_d)$ belongs to $\overset{\circ}{D}_\Lambda$, where O_d denotes the $d \times d$ matrix whose coefficients are all zero. As a consequence Cramér's theorem (cf. [3]) implies that $(\tilde{\nu}_{n,\rho})_{n \geq 1}$ satisfies the large deviations principle with speed n and governed by I .

3 Convergence in probability of $(S_n/n, T_n/n)$

We saw in the previous section that, under the hypothesis of theorem 1, the sequence $(\tilde{\nu}_{n,\rho})_{n \geq 1}$ satisfies the large deviations principle with speed n and governed by I . This and Varadhan's lemma (see [3]) suggest that, asymptotically, $(S_n/n, T_n/n)$ concentrates on the minima of the function $I - F$. In subsection 3.a), we prove that $I - F$ has a unique minimum at $(0, \Sigma)$ on Δ^* and we extend F on the entire closed set Δ so that it remains true on Δ . This is the key ingredient for the proof of the law of large numbers in theorem 1, given in subsection 3.b).

a) Minimum de $I - F$

Proposition 5. *If ρ is a symmetric non-degenerate probability measure on \mathbb{R}^d , then*

$$\forall x \in \mathbb{R}^d \setminus \{0\} \quad \forall M \in \mathcal{S}_d^{++} \quad I(x, M) > \frac{\langle M^{-1}x, x \rangle}{2}.$$

Moreover, if Λ is finite in a neighbourhood of $(0, O_d)$, then the function $I - F$ has a unique minimum at $(0, \Sigma)$ on Δ^ .*

Proof. Let $x \in \mathbb{R}^d \setminus \{0\}$ and $M \in \mathcal{S}_d^{++}$. By taking $A = -M^{-1}x {}^t x M^{-1}/2$ and $u = M^{-1}x$, we get

$$\langle u, x \rangle + \text{tr}(AM) = \langle M^{-1}x, x \rangle - \frac{1}{2} \text{tr}(M^{-1}x {}^t x) = \frac{\langle M^{-1}x, x \rangle}{2}.$$

As a consequence

$$I(x, M) \geq \frac{\langle M^{-1}x, x \rangle}{2} - \Lambda \left(M^{-1}x, -\frac{1}{2}M^{-1}x {}^t x M^{-1} \right).$$

For any $z \in \mathbb{R}^d$, we have ${}^t z M^{-1}x = \langle M^{-1}x, z \rangle = \text{tr}(z {}^t (M^{-1}x)) \in \mathbb{R}$ thus

$$-\frac{1}{2} \text{tr}(z {}^t z M^{-1}x {}^t x M^{-1}) = -\frac{\langle M^{-1}x, z \rangle}{2} \text{tr}(z {}^t x M^{-1}) = -\frac{\langle M^{-1}x, z \rangle^2}{2}.$$

Therefore

$$\Lambda \left(M^{-1}x, -\frac{1}{2}M^{-1}x {}^t x M^{-1} \right) = \ln \int_{\mathbb{R}^d} \exp \left(\langle M^{-1}x, z \rangle - \frac{\langle M^{-1}x, z \rangle^2}{2} \right) d\rho(z).$$

By symmetry of ρ , we have, for any $s \in \mathbb{R}^d$,

$$\begin{aligned} \int_{\mathbb{R}^d} \exp \left(\langle s, z \rangle - \frac{\langle s, z \rangle^2}{2} \right) d\rho(z) &= \int_{\mathbb{R}^d} \exp \left(-\langle s, z \rangle - \frac{\langle s, z \rangle^2}{2} \right) d\rho(z) \\ &= \frac{1}{2} \left(\int_{\mathbb{R}^d} \exp \left(\langle s, z \rangle - \frac{\langle s, z \rangle^2}{2} \right) d\rho(z) + \int_{\mathbb{R}^d} \exp \left(-\langle s, z \rangle - \frac{\langle s, z \rangle^2}{2} \right) d\rho(z) \right) \\ &= \int_{\mathbb{R}^d} \cosh(\langle s, z \rangle) \exp \left(-\frac{\langle s, z \rangle^2}{2} \right) d\rho(z). \end{aligned}$$

As a consequence

$$\begin{aligned} \Lambda \left(M^{-1}x, -\frac{1}{2}M^{-1}x {}^t x M^{-1} \right) &= \\ &= \ln \int_{\mathbb{R}^d} \cosh(\langle M^{-1}x, z \rangle) \exp \left(-\frac{\langle M^{-1}x, z \rangle^2}{2} \right) d\rho(z). \end{aligned}$$

It is straightforward to see that the function $y \mapsto 1 - \cosh(y) \exp(-y^2/2)$ is non-negative on \mathbb{R} and vanishes only at 0. Hence, for any $z \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \cosh(\langle s, z \rangle) \exp \left(-\frac{\langle s, z \rangle^2}{2} \right) d\rho(z) \leq 1,$$

and equality holds if and only if $\rho(\{z : \langle s, z \rangle = 0\}) = 1$. The non-degeneracy of ρ implied that the equality case only holds if $s = 0$. Applying this to $s = M^{-1}x \neq 0$, we obtain

$$\Lambda \left(M^{-1}x, -\frac{1}{2}M^{-1}x {}^t x M^{-1} \right) < 0,$$

and thus $I(x, M) > \langle M^{-1}x, x \rangle / 2$.

Suppose now that $x = 0$ and $M \in \mathcal{S}_d^{++}$. Then

$$I(x, M) - \frac{\langle M^{-1}x, x \rangle}{2} = I(0, M).$$

If we assume that Λ is finite in a neighbourhood of $(0, \dots, 0, O_d)$, then $I(0, M) = 0$ if and only if $M = \Sigma$ (see proposition III.4 of [8]). This ends the proof of the lemma. \square

However, in order to apply Varadhan's lemma, F must be extended to an upper semi-continuous function on the entire closed set Δ . To this end, we put

$$\forall (x, M) \in \Delta \setminus \Delta^* \quad F(x, M) = \frac{1}{2},$$

and it is easy to check that F is indeed an upper semi-continuous function on Δ .

Now we investigate on how the inequality in proposition 5 holds on Δ .

Let $(x, M) \in \mathbb{R}^d \times \mathcal{S}_d^+$. We denote by $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_d$ the eigenvalues (not necessary distinct) of M . There exists an orthogonal matrix P such that $M = PD {}^t P$, where D is the diagonal matrix such that $D_{i,i} = \lambda_i$ for any $i \in \{1, \dots, d\}$. We have

$$\begin{aligned} I(x, M) &= \sup_{(u, A) \in \mathbb{R}^d \times \mathcal{S}_d} \left(\langle x, u \rangle + \text{tr}(PD {}^t P A) - \Lambda(u, A) \right) \\ &= \sup_{(u, A) \in \mathbb{R}^d \times \mathcal{S}_d} \left(\langle x, u \rangle + \text{tr}(DA) - \Lambda(u, PA {}^t P) \right). \end{aligned}$$

Assume that $M \notin \mathcal{S}_d^{++}$ and denote by $k = k_M \geq 1$ the dimension of the kernel of M . Let $a \in]-\infty, 0[$. By taking $u = 0$ and A the symmetric matrix such that

$$\forall (i, j) \in \{1, \dots, d\} \quad A_{i,j} = \begin{cases} a & \text{if } i = j \in \{1, \dots, k\}, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$I(x, M) \geq -\Lambda(u, PA {}^t P) = -\ln \int_{\mathbb{R}^d} \exp \langle PA {}^t P z, z \rangle d\rho(z),$$

i.e.,

$$\forall a \in \mathbb{R} \quad I(x, M) \geq -\ln \int_{\mathbb{R}^d} \exp \left(a \sum_{j=1}^k ({}^t P z)_j^2 \right) d\rho(z).$$

For any $z \in \mathbb{R}^d$, we have

$$\begin{aligned} \sum_{j=1}^k ({}^tPz)_j^2 = 0 &\iff \forall j \in \{1, \dots, k\} \quad \langle Pe_j, z \rangle = \langle e_j, {}^tPz \rangle = ({}^tPz)_j = 0 \\ &\iff \forall x \in \text{Ker}(M) \quad \langle x, z \rangle = 0 \\ &\iff z \in \text{Ker}(M)^\perp, \end{aligned}$$

since (Pe_1, \dots, Pe_k) is a basis of $\text{Ker}(M)$ (they are the eigenvectors of M associated to the eigenvalue 0). As a consequence

$$\forall z \in \mathbb{R}^d \quad \exp \left(a \sum_{j=1}^k ({}^tPz)_j^2 \right) \xrightarrow{a \rightarrow -\infty} \mathbb{1}_{\text{Ker}(M)^\perp}(z).$$

Moreover the left term defines a function which is bounded above by 1. Therefore the dominated convergence theorem implies that

$$\int_{\mathbb{R}^d} \exp \left(a \sum_{j=1}^k ({}^tPz)_j^2 \right) d\rho(z) \xrightarrow{a \rightarrow -\infty} \rho(\text{Ker}(M)^\perp).$$

Whence

$$I(x, M) \geq -\ln \rho(\text{Ker}(M)^\perp).$$

So that $I(x, M) > 1/2$, it is enough to have $\rho(\text{Ker}(M)^\perp) < e^{-1/2}$. Since $\text{Ker}(M)^\perp$ is included in some vector hyperplane of \mathbb{R}^d , we obtain the following proposition:

Proposition 6. *Let ρ be a symmetric probability measure on \mathbb{R}^d satisfying (*). Suppose that the ρ -measure of any vector hyperplane of \mathbb{R}^d is less than $1/\sqrt{e}$. Then $I - F$ has a unique minimum at $(0, \Sigma)$ on Δ .*

b) Convergence of $(S_n/n, T_n/n)$ under $\tilde{\mu}_{n,\rho}$

Let us first prove the following proposition, which is a consequence of Varadhan's lemma.

Proposition 7. *Let ρ be a symmetric probability measure on \mathbb{R}^d with a positive definite covariance matrix Σ . We have*

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \ln Z_n \geq 0.$$

Suppose that ρ satisfies () and that the ρ -measure of any vector hyperplane of \mathbb{R}^d is less than $1/\sqrt{e}$. If \mathcal{A} is a closed subset of $\mathbb{R}^d \times \mathcal{S}_d$ which does not contain $(0, \Sigma)$, then*

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^* \cap \mathcal{A}} \exp \left(\frac{n \langle M^{-1}x, x \rangle}{2} \right) d\tilde{\mu}_{n,\rho}(x, M) < 0.$$

Proof. The set $\overset{\circ}{\Delta}$, the interior of Δ^* , contains $(0, \Sigma)$ thus the large deviations principle satisfied by $(\tilde{\nu}_{n,\rho})_{n \geq 1}$ implies that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln Z_n &= \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^*} \exp \left(\frac{n \langle M^{-1}x, x \rangle}{2} \right) d\tilde{\nu}_{n,\rho}(x, M) \\ &\geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \tilde{\nu}_{n,\rho}(\Delta^*) \geq -\inf \left\{ I(x, M) : (x, M) \in \overset{\circ}{\Delta} \right\} = 0. \end{aligned}$$

We prove now the second inequality. Since ρ verifies la condition $(*)$, we have that $(0, O_d) \in \overset{\circ}{D}_\Lambda$. Cramér's theorem (cf. [3]) implies then that $(\tilde{\nu}_{n,\rho})_{n \geq 1}$ satisfies the large deviations principle with speed n and governed by the good rate function I . Since F is upper semi-continuous on the closed set Δ , Varadhan's lemma (cf. [3]) yields

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^* \cap \mathcal{A}} \exp \left(\frac{n \langle M^{-1}x, x \rangle}{2} \right) d\tilde{\nu}_{n,\rho}(x, M) \\ \leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta \cap \mathcal{A}} \exp(nF(x, M)) d\tilde{\nu}_{n,\rho}(x, M) \leq \sup_{\Delta \cap \mathcal{A}} (F - I). \end{aligned}$$

Since the ρ -measure of any vector hyperplane of \mathbb{R}^d is less than $1/\sqrt{e}$, proposition 6 implies that $I - F$ has a unique minimum at $(0, \Sigma)$ on Δ . Since the closed subset $\Delta \cap \mathcal{A}$ does not contain $(0, \Sigma)$ and since F is upper semi-continuous and I is a good rate function, we have

$$\sup_{\Delta \cap \mathcal{A}} (F - I) < 0.$$

This proves the second inequality of the proposition. \square

Proof of the law of large numbers in theorem 1. Suppose that ρ is symmetric, satisfies $(*)$ and that the ρ -measure of any vector hyperplane of \mathbb{R}^d is less than $1/\sqrt{e}$. Let us denote by $\theta_{n,\rho}$ the law of $(S_n/n, T_n/n)$ under $\tilde{\mu}_{n,\rho}$. Let U be an open neighbourhood of $(0, \Sigma)$ in $\mathbb{R}^d \times \mathcal{S}_d$. Proposition 7 implies that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \theta_{n,\rho}(U^c) &= \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^* \cap U^c} \exp \left(\frac{n \langle M^{-1}x, x \rangle}{2} \right) d\tilde{\nu}_{n,\rho}(x, M) \\ &\quad - \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln Z_n < 0. \end{aligned}$$

Hence there exist $\varepsilon > 0$ and $n_0 \geq 1$ such that $\theta_{n,\rho}(U^c) \leq e^{-n\varepsilon}$ for any $n \geq n_0$. Therefore, for any neighbourhood U of $(0, \Sigma)$,

$$\lim_{n \rightarrow +\infty} \tilde{\mu}_{n,\rho} \left(\left(\frac{S_n}{n}, \frac{T_n}{n} \right) \in U^c \right) = 0,$$

i.e., under $\tilde{\mu}_{n,\rho}$, $(S_n/n, T_n/n)$ converges in probability to $(0, \Sigma)$. \square

4 Convergence in distribution of $T_n^{-1/2} S_n/n^{1/4}$ under $\tilde{\mu}_{n,\rho}$

In this section, we generalize theorem 1 of [9] to the higher dimension in order to prove our fluctuation result.

Theorem 8. *Let ρ be a symmetric non-degenerate probability measure on \mathbb{R}^d such that*

$$\int_{\mathbb{R}^d} \|z\|^5 d\rho(z) < +\infty.$$

Let Σ the covariance matrix of ρ and let M_4 be the function defined in theorem 1. Then, under $\tilde{\mu}_{n,\rho}$,

$$\frac{1}{n^{1/4}} T_n^{-1/2} S_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{\exp\left(-\frac{1}{12} M_4(\Sigma^{-1/2} z)\right) dz}{\int_{\mathbb{R}^d} \exp\left(-\frac{1}{12} M_4(\Sigma^{-1/2} u)\right) du}.$$

In the proof of this theorem, we show that the limiting law is well defined. Notice that, if $d = 1$, then $\Sigma^{-1/2} = \sigma^{-1}$ and

$$\forall z \in \mathbb{R} \quad M_4(\Sigma^{-1/2} z) = \frac{\mu_4 z^4}{\sigma^4}.$$

Hence theorem 8 is indeed a generalization of theorem 1 of [9]

a) Proof of theorem 8

Let $(X_n^k)_{n \geq d, 1 \leq k \leq n}$ be an infinite triangular array of random variables such that, for any $n \geq 1$, (X_n^1, \dots, X_n^n) has the law $\tilde{\mu}_{n,\rho}$. Let us recall that

$$\forall n \geq 1 \quad S_n = X_n^1 + \dots + X_n^n \quad \text{and} \quad T_n = X_n^1 \text{ }^t(X_n^1) + \dots + X_n^n \text{ }^t(X_n^n).$$

and that $T_n \in \mathcal{S}_d^{++}$ almost surely. We use the Hubbard-Stratonovich transformation: let W be a random vector with standard multivariate Gaussian distribution and which is independent of $(X_n^k)_{n \geq d, 1 \leq k \leq n}$. Let $n \geq 1$ and f be a bounded continuous function on \mathbb{R}^d . We put

$$E_n = \mathbb{E} \left[f \left(\frac{W}{n^{1/4}} + \frac{1}{n^{1/4}} T_n^{-1/2} S_n \right) \right].$$

We introduce $(Y_i)_{i \geq 1}$ a sequence of independent random vectors with common distribution ρ . We denote

$$A_n = \sum_{i=1}^n Y_i, \quad B_n = \left(\sum_{i=1}^n Y_i \text{ }^t Y_i \right)^{1/2} \quad \text{and} \quad \mathcal{B}_n = \{\det(B_n^2) > 0\}.$$

We have

$$E_n = \frac{1}{Z_n (2\pi)^{d/2}} \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f \left(\frac{w}{n^{1/4}} + \frac{1}{n^{1/4}} B_n^{-1} A_n \right) \times \exp \left(\frac{1}{2} \left\langle B_n^{-2} A_n, A_n \right\rangle - \frac{\|w\|^2}{2} \right) dw \right].$$

We make the change of variables $z = n^{-1/4}(w + B_n^{-1}A_n)$ in the integral and we get

$$E_n = C_n \mathbb{E} \left[\mathbf{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{\sqrt{n}\|z\|^2}{2} + n^{1/4} \langle z, B_n^{-1} A_n \rangle \right) dz \right]$$

where $C_n = n^{d/4} Z_n^{-1} (2\pi)^{-d/2}$. Let $U_1, \dots, U_n, \varepsilon_1, \dots, \varepsilon_n$ be independent random variables such that the distribution of U_i is ρ and the distribution of ε_i is $(\delta_{-1} + \delta_1)/2$, for any $i \in \{1, \dots, n\}$. Since ρ is symmetric, the random variables $\varepsilon_1 U_1, \dots, \varepsilon_n U_n$ are also independent with common distribution ρ . Therefore

$$E_n = C_n \mathbb{E} \left[\mathbf{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{\sqrt{n}\|z\|^2}{2} + n^{1/4} \left\langle z, B_n^{-1} \left(\sum_{i=1}^n \varepsilon_i U_i \right) \right\rangle \right) dz \right].$$

In the case where the matrix $B_n^2 = U_1 {}^t U_1 + \dots + U_n {}^t U_n$ is invertible, we denote

$$\forall i \in \{1, \dots, n\} \quad a_{i,n} = \left(\sum_{j=1}^n U_j {}^t U_j \right)^{-1/2} U_i.$$

By using Fubini's lemma and the independence of $\varepsilon_i, U_i, i \geq 1$, we obtain

$$E_n = C_n \mathbb{E} \left[\mathbf{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{\sqrt{n}\|z\|^2}{2} \right) \times \mathbb{E} \left(\prod_{i=1}^n \exp \left(n^{1/4} \varepsilon_i \langle z, a_{i,n} \rangle \right) \middle| (U_1, \dots, U_n) \right) dz \right].$$

Therefore

$$E_n = C_n \mathbb{E} \left[\mathbf{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{\sqrt{n}\|z\|^2}{2} \right) \exp \left(\sum_{i=1}^n \ln \cosh (n^{1/4} \langle z, a_{i,n} \rangle) \right) dz \right].$$

We define the function g by

$$\forall y \in \mathbb{R} \quad g(y) = \ln \cosh y - \frac{y^2}{2}.$$

It is easy to see that $g(y) < 0$ if $y > 0$. We notice that, for any x and y in \mathbb{R}^d , $\langle x, y \rangle^2 = \langle x, (y {}^t y) x \rangle$. Therefore

$$\sum_{i=1}^n \langle z, a_{i,n} \rangle^2 = \sum_{i=1}^n \langle z, (a_{i,n} {}^t a_{i,n}) z \rangle = \left\langle z, \left(\sum_{i=1}^n a_{i,n} {}^t a_{i,n} \right) z \right\rangle = \langle z, I_d z \rangle = \|z\|^2.$$

As a consequence

$$E_n = C_n \mathbb{E} \left[\mathbf{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) dz \right].$$

Now we use Laplace's method. Let us examine the convergence of the term in the exponential: for any $z \in \mathbb{R}^d$ and $i \in \{1, \dots, n\}$, the Taylor-Lagrange formula states that there exists a random variable $\xi_{n,i}$ such that

$$g(n^{1/4} \langle z, a_{i,n} \rangle) = -\frac{n \langle z, a_{i,n} \rangle^4}{12} + \frac{n^{3/2} \langle z, a_{i,n} \rangle^5}{n^{1/4} 5!} g^{(5)}(\xi_{n,i}).$$

Let $z \in \mathbb{R}^d$. We have

$$n \sum_{i=1}^n \langle z, a_{i,n} \rangle^4 = n \sum_{i=1}^n \langle B_n^{-1} z, U_i \rangle^4 = \frac{1}{n} \sum_{i=1}^n \langle \sqrt{n} B_n^{-1} z, U_i \rangle^4.$$

We denote $\zeta_n = \sqrt{n} B_n^{-1} z$. We have

$$\begin{aligned} n \sum_{i=1}^n \langle z, a_{i,n} \rangle^4 &= \frac{1}{n} \sum_{i=1}^n \langle \zeta_n, U_i \rangle^4 = \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^d (\zeta_n)_j (U_i)_j \right)^4 \\ &= \sum_{1 \leq j_1, j_2, j_3, j_4 \leq d} (\zeta_n)_{j_1} (\zeta_n)_{j_2} (\zeta_n)_{j_3} (\zeta_n)_{j_4} \frac{1}{n} \sum_{i=1}^n (U_i)_{j_1} (U_i)_{j_2} (U_i)_{j_3} (U_i)_{j_4}. \end{aligned}$$

Since ρ is non-degenerate, its covariance matrix Σ is invertible. Moreover ρ has a finite fourth moment thus the law of large number implies that

$$\zeta_n \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \Sigma^{-1/2} z,$$

and that, for any $(j_1, j_2, j_3, j_4) \in \{1, \dots, d\}^4$,

$$\frac{1}{n} \sum_{i=1}^n (U_i)_{j_1} (U_i)_{j_2} (U_i)_{j_3} (U_i)_{j_4} \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \int_{\mathbb{R}^d} y_{j_1} y_{j_2} y_{j_3} y_{j_4} d\rho(y).$$

As a consequence

$$n \sum_{i=1}^n \langle z, a_{i,n} \rangle^4 \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} M_4 \left(\Sigma^{-1/2} z \right).$$

Since ρ has a finite fifth moment, we prove similarly that

$$n^{3/2} \sum_{i=1}^n \langle z, a_{i,n} \rangle^5 \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} M_5 \left(\Sigma^{-1/2} z \right),$$

where, for any $z \in \mathbb{R}^d$,

$$M_5(z) = \sum_{1 \leq j_1, j_2, j_3, j_4, j_5 \leq d} \left(\int_{\mathbb{R}^d} y_{j_1} y_{j_2} y_{j_3} y_{j_4} y_{j_5} d\rho(y) \right) z_{j_1} z_{j_2} z_{j_3} z_{j_4} z_{j_5}.$$

Finally, by a simple computation, we see that $g^{(5)}$ is bounded over \mathbb{R} . Hence

$$\forall z \in \mathbb{R}^d \quad \sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} -\frac{1}{12} M_4 \left(\Sigma^{-1/2} z \right).$$

Lemma 9. *There exists $c > 0$ such that*

$$\forall z \in \mathbb{R}^d \quad \forall n \geq 1 \quad \sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \leq -\frac{c \|z\|^4}{1 + \|z\|^2 / \sqrt{n}}.$$

The proof of this lemma follows the same lines than the proof of lemma 3 in [9].

If $\|z\| \leq n^{1/4}$ then $1 + \|z\|^2/\sqrt{n} \leq 2$ and thus, by the previous lemma,

$$\left| \mathbb{1}_{\mathcal{B}_n} \mathbb{1}_{\|z\| \leq n^{1/4}} \exp \left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) \right| \leq \exp \left(-\frac{c\|z\|^4}{2} \right).$$

Thus the dominated convergence theorem implies that

$$z \mapsto \exp \left(-M_4 \left(\Sigma^{-1/2} z \right) / 12 \right)$$

is integrable on \mathbb{R}^d and that

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} \mathbb{1}_{\|z\| \leq n^{1/4}} f(z) \exp \left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) dz \right] \\ \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{1}{12} M_4 \left(\Sigma^{-1/2} z \right) \right) dz. \end{aligned}$$

If $\|z\| > n^{1/4}$ then $1 + \|z\|^2/\sqrt{n} \leq 2\|z\|^2/\sqrt{n}$ and thus, by the previous lemma,

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} \mathbb{1}_{\|z\| > n^{1/4}} f(z) \exp \left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) dz \right] \\ \leq \|f\|_{\infty} \int_{\mathbb{R}^d} \exp \left(-\frac{c\sqrt{n}\|z\|^2}{2} \right) dz = \frac{\|f\|_{\infty} (2\pi)^{d/2}}{n^{d/4} c^{d/2}} \xrightarrow{n \rightarrow +\infty} 0, \end{aligned}$$

and thus

$$\begin{aligned} \frac{E_n}{C_n} = \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) dz \right] \\ \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{1}{12} M_4 \left(\Sigma^{-1/2} z \right) \right) dz. \end{aligned}$$

If we take $f = 1$, we get

$$\frac{1}{C_n} = \frac{Z_n (2\pi)^{d/2}}{n^{d/4}} \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} \exp \left(-\frac{1}{12} M_4 \left(\Sigma^{-1/2} z \right) \right) dz.$$

By Paul Levy's theorem, we have then

$$\frac{W}{n^{1/4}} + \frac{1}{n^{1/4}} T_n^{-1/2} S_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{\exp \left(-\frac{1}{12} M_4(\Sigma^{-1/2} z) \right) dz}{\int_{\mathbb{R}^d} \exp \left(-\frac{1}{12} M_4(\Sigma^{-1/2} u) \right) du}.$$

Since $(W n^{-1/4})_{n \geq 1}$ converges in distribution to 0, Slutsky lemma (theorem 3.9 of [1]) implies the convergence in distribution of theorem 8.

We remark that the hypothesis that ρ has a finite fifth moment may certainly be weakened by assuming instead that

$$\exists \varepsilon > 0 \quad \int_{\mathbb{R}^d} \|z\|^{4+\varepsilon} d\rho(z) < +\infty.$$

b) Proof of the fluctuation result in theorem 1.

In section 3, we proved the law of large numbers in theorem 1. It implies that, under $\tilde{\mu}_{n,\rho}$, T_n/n converges in probability to Σ . Moreover the hypothesis (*) implies that $(0, O_d) \in \overset{\circ}{D}_\Lambda$ and thus ρ has finite moments of all orders. Theorem 8 and Slutsky lemma yield

$$\frac{S_n}{n^{3/4}} = \left(\frac{T_n}{n}\right)^{1/2} \times \frac{1}{n^{1/4}} T_n^{-1/2} S_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{\exp\left(-\frac{1}{12}M_4(\Sigma^{-1}z)\right) dz}{\int_{\mathbb{R}^d} \exp\left(-\frac{1}{12}M_4(\Sigma^{-1}u)\right) du}.$$

Theorem 1 is proved.

References

- [1] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., second edition, 1999. A Wiley-Interscience Publication.
- [2] Raphaël Cerf and Matthias Gorny. A Curie–Weiss model of self–organized criticality. *Ann. Probab.*, to appear, 2015.
- [3] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, 2010.
- [4] Richard S. Ellis and Charles M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. Verw. Gebiete*, 44(2):117–139, 1978.
- [5] Matthias Gorny. A Curie–Weiss model of self–organized criticality: the Gaussian case. *Mark. Proc. Rel. Fields*, 20(3):563–576, 2014.
- [6] Matthias Gorny. A Dynamical Curie–Weiss model of SOC: the Gaussian case. to appear, 2015.
- [7] Matthias Gorny. The Cramér condition for the Curie–Weiss model of SOC. *Braz. J. Probab. Stat.*, to appear, 2015.
- [8] Matthias Gorny. *Un modèle d’Ising Curie–Weiss de criticalité auto-organisée*. Phd thesis, Université Paris-Sud, 2015. <https://tel.archives-ouvertes.fr/tel-01167487>.
- [9] Matthias Gorny and S. R. Srinivasa Varadhan. Fluctuations of the self-normalized sum in the Curie-Weiss model of SOC. *J. Stat. Phys.*, 160(3):513–518, 2015.

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